Exponential Functions

Exponential Functions

- An **exponential function** with **base *b*** is denoted by $\boldsymbol{f(x) = b^x}$ where *b* and *x* are any real numbers such that $b > 0$ and $b \neq 1$. Review sections 0.2-0.3 for properties of exponents.
- **Example 1:** Let $f(x) = 4^x, h(x) = \frac{1}{9}^x, g(x) = 10^{x-1}$. Find the following values. If an approximation is required, approximate to four decimal places.

$$f(2), f(\pi), h\left(-\frac{3}{2}\right), g(2.3), f(0), h(0)$$

$$f(2) = 4^2 = 16$$

$$f(\pi) = 4^\pi \approx 77.8802$$

$$h\left(-\frac{3}{2}\right) = \frac{1}{9}^{-3/2} = 9^{3/2} = \sqrt{9}^3 = 3^3 = 27$$

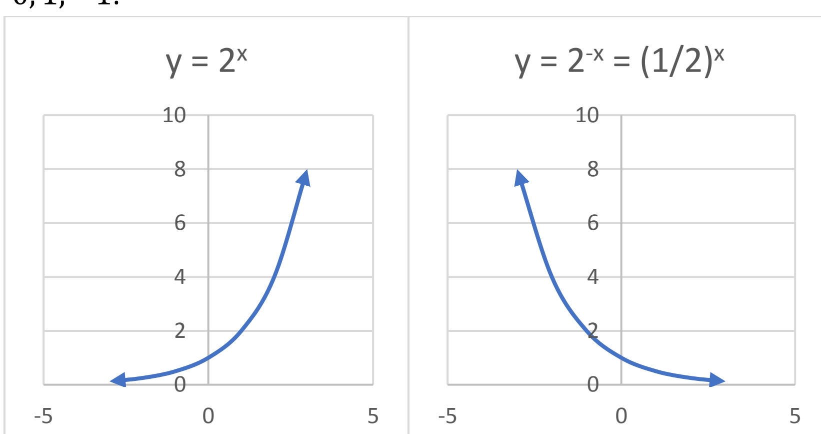
$$g(2.3) = 10^{2.3-1} = 10^{1.3} \approx 19.9526$$

$$f(0) = 4^0 = 1$$

$$h(0) = \frac{1}{9}^0 = 1$$

Graphs of Exponential Functions

- Exponential functions can be graphed by plotting points. Usually, it is useful to find the points for $x = 0, 1, -1$.



- **Example 1:** Graph the function $f(x) = 4^x$.
  Label the y-intercept by finding $f(x) = 4^x$ when $x = 0$.

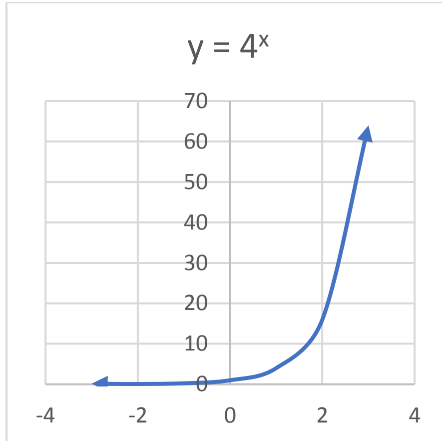
$$f(0) = 4^0 = 1$$

  First point is (0, 1).
  Then find $f(1)$ and $f(-1)$.

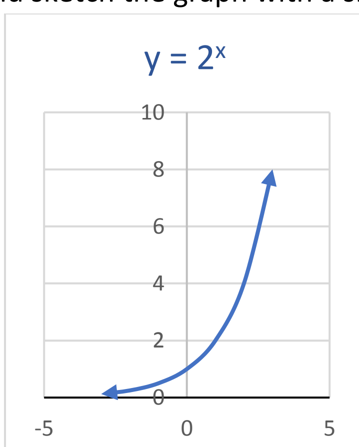
$$f(1) = 4^1 = 4$$

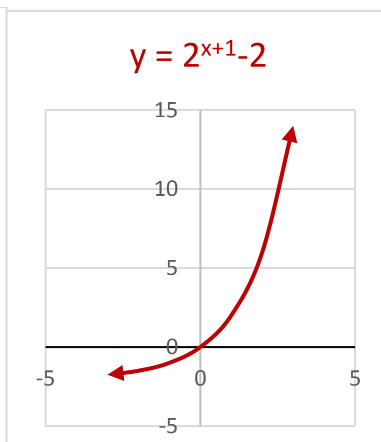
$$f(-1) = 4^{-1} = 0.25$$

Plot the points and then sketch the curve with a horizontal asymptote.



- **Example 2:** Graph the function $f(x) = 2^{x+1} - 2$. State the domain and range of the function.
  First, identify the base function: $f(x) = 2^x$
  Identify the base function y-intercept and horizontal asymptote: (0, 1) and y = 0.
  Since a 1 is added to x, the graph is shifted one unit to the left.
  Since a 2 is subtracted from 2^{x+1}, the graph is shifted two units down.
  Shift the y-intercept from (0, 1) to (0-1, 1-2) = (-1, -1).
  Shift the horizontal asymptote down two units from y = 0 to y = -2.
  Find additional points on the graph. $f(0) = 2^{0+1} - 2 = 2 - 2 = 0$
  $f(1) = 2^{1+1} - 2 = 4 - 2 = 2$
  Plot the points and sketch the graph with a smooth curve.



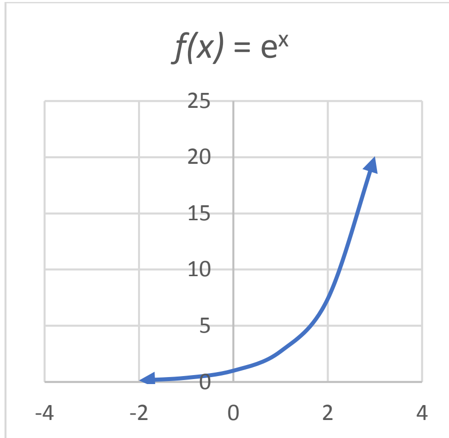


The domain of the function is $(-\infty, \infty)$.
The range of the function is $(-2, \infty)$.

The Natural Base e

- The irrational number e appears in many applications and is called the natural base. The exponential function with base e $f(x) = e^x$ is called the exponential function or the natural exponential function.

- $e \approx 2.71828$



- **Example 1:** Graph the function $f(x) = 3 + e^{2x}$.
  First, create a table with points to plot on the graph.

x	$f(x) = 3 + e^{2x}$	(x, y)
-2	3.02	$(-2, 3.02)$
-1	3.14	$(-1, 3.14)$
0	4	$(0, 4)$
1	10.39	$(1, 10.39)$
2	57.60	$(2, 57.60)$

Note: These values need to be found using a calculator and will need to be rounded.

Applications

- Exponential functions describe either growth or decay.
- **Example 1**: Doubling Time of Populations
  Use the doubling time growth model:

$$P = P_0 2^{t/d}$$

P is the population at time t.
P_0 is the population at time t = 0.
d is the doubling time.

The current population of an island is 800,000, and the population is expected to double in 20 years. Estimate the population in 4 years. Round your answer to the nearest thousand.

$$P_0 = 800000, t = 4, d = 20$$

$$P = P_0 2^{t/d} = (800000)\left(2^{\frac{4}{20}}\right) = (800000)(2^{\frac{1}{5}}) \approx 918959$$

In 4 years, there will be approximately 919,000 people on the island.

- **Example 2**: Radioactive Decay: half-life

Use the half-life model:

$$A = A_0(\frac{1}{2})^{t/h}$$

The radioactive isotope of potassium which is used in the diagnosis of brain tumors, has a half-life of 12.36 hours. If 700 milligrams of this potassium are taken, how many milligrams will remain after 48 hours? Round to the nearest milligram.

$$A_0 = 700, t = 48, h = 12.36$$

$$A = 700(\frac{1}{2})^{48/12.36} \approx 700(0.0678) \approx 47.46$$

After 48 hours, there are approximately 47 milligrams of potassium left.

- **Example 3:** Compound Interest

  If a principal P is invested at an annual rate r compounded n times a year, then the amount A in the account at the end of t years is given by

$$A = P(1 + \frac{r}{n})^{nt}$$

The annual interest rate r is expressed as a decimal.

Typical Number of Times Interest Is Compounded	
Annually	$n = 1$
Semiannually	$n = 2$
Quarterly	$n = 4$
Monthly	$n = 12$
Weekly	$n = 52$
Daily	$n = 356$

If $4500 is deposited in an account paying 4% compounded monthly, how much will you have in the account in 8 years?

$$P = 4500, r = 0.04, n = 12, t = 8$$

$$A = 4500(1 + \frac{0.04}{12})^{12*8} = 4500(1 + \frac{0.04}{12})^{96} \approx 6193.78$$

You will have approximately $6193.78 in the account.

- **Example 4:** Continuous Compound Interest

  If a **principal *P*** is invested at an annual **rate *r* compounded continuously**, then the **amount *A*** in the account at the end of *t* years is given by

$$A = Pe^{rt}$$

The annual interest rate *r* is expressed as a decimal.

If $5000 is deposited in a savings account paying 3.5% a year compounded continuously, how much will you have in the account in 8 years?

$$P = 5000, r = 0.035, t = 8$$

$$A = (5000)e^{0.035*8} = (5000)e^{0.28} \approx 6615.65$$

There will be $6615.65 in the account in 8 years.

Logarithmic Functions

Logarithmic Functions

- For $x > 0$, $b > 0$, and $b \neq 1$, the **logarithmic function** with **base *b*** is denoted by $\boldsymbol{f(x) = \log_b x}$ where

$$\boldsymbol{y = \log_b x} \textbf{ if and only if } \boldsymbol{x = b^y}$$

Read "log base b of x"

- **Example 1:** Rewrite the following logarithms in exponential form using

$$y = \log_b x \text{ if and only if } x = b^y$$

Where b, the base, is represented in green, x, the information within our logarithm and the solution in our exponential, is represented in blue, and y, the solution to our logarithm and the exponent in our exponential is represented in pink.

a) $\log_3 9 = 2$

Exponential form: $9 = 3^2$

b) $\log_5 125 = 3$

Exponential form: $125 = 5^3$

c) $\log_{64} 8 = \frac{1}{2}$

Exponential form: $8 = 64^{\frac{1}{2}}$

- **Example 2:** Rewrite the following exponentials in logarithmic form using

$$y = \log_b x \text{ if and only if } x = b^y$$

Where b, the base, is represented in green, x, the information within our logarithm and the solution in our exponential, is represented in blue, and y, the solution to our logarithm and the exponent in our exponential is represented in pink.

a) $216 = 6^3$

Logarithmic form: $\log_3 9 = 2$

b) $125 = 5^3$

Logarithmic form: $\log_3 9 = 2$

c) $125 = 5^3$

Logarithmic form: $\log_3 9 = 2$

- **Example 3:** Evaluate the exact value of the following logarithm:

$$\log_2 8 = ?$$

Step 1: Figure out the base of the exponent.

$$\log_2 8 = ?$$

Our base in this problem is 2

Step 2: Ask yourself "2 *to what power* will give me 8?"

We know that 2 to the power of 3 is 8

Step 3: Change the logarithm into exponential form

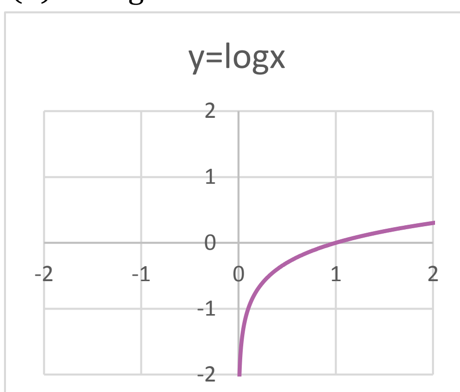
$$2^3 = 8$$

Common and Natural Logarithms

- The bases 10 and e are 2 of the most common logarithmic functions. Because they are common, we rewrite the logarithm in a simpler way.
  - Instead of $\mathbf{\log_{10} x}$, you will most likely see it written as $\mathbf{\log x}$**.** In other words, if there is no base written on your logarithm, you may assume it is base 10
  - Instead of $\mathbf{\log_e x}$, you will most likely see it written as $\mathbf{\ln x}$. In other words, $\ln$ is another way to write a logarithm with base e.

Graphs of Logarithmic Functions

- Below is the graph of $f(x) = \log x$



- **Interpreting the graph:** To begin interpreting the graph, let's take a look at a few major points.

x	f(x)	Importance
1	0	This tells us that f(x) has an x-intercept at (1,0)
1000	3	This shows that as x gets further away from its x-intercept, the y-values increase slowly.
$\frac{1}{1000}$	-100	This shows us as x gets closer and closer to 0, our y-values decrease rapidly.

- **Key features of the graph:**
  - $f(x) = \log x$ has a vertical asymptote at x = 0 (the y-axis).
  - Negative x-values *cannot* be evaluated in the function $f(x) = \log x$. They *do not exist*.
  - The domain of a logarithmic function is $(0, \infty)$.
  - The range of a logarithmic function is $(-\infty, \infty)$.

Applications

- A **decibel** can be defined as

$$D = 10 \log \frac{I}{I_T}$$

Where D is decibel level (dB), I is the measure of intensity (watts per square meter), and I_T is the intensity threshold of the least audible sound a human is able to hear. In further problems, we will use $I_T = 1 \times 10^{-12}\ \frac{\text{W}}{\text{m}^2}$.

- **Example 1:** Calculate the decibel level associated with the typical sound intensity of a rock band playing with intensity of $I = 1\ \times 10^{-1}$.

$$D = 10 \log \frac{I}{I_T}$$

$$D = 10 \log \frac{1\ \times 10^{-1}}{1 \times 10^{-12}}$$

$$D = 10 \log(1\ \times 10^{11})$$

$$D = 10 \log(10^{11})$$

$$D = 10\ \cdot 11$$

$$D = 110\ \frac{\text{W}}{\text{m}^2}$$

- The **Richter scale** is used to determine the magnitude of an earth quake. Its equation is given by:

$$M = \frac{2}{3} \log \frac{E}{E_0}$$

Where M is the magnitude, E is the seismic energy released by the earthquake (in joules) and E_0 is the energy released by a reference earthquakes ($E_0 = 10^{4.4}$ joules).

- **Example 2:** 1.23 Using the Richter scale, what is the magnitude of an earthquake that released 1.27×10^{15} joules of seismic energy.

$$M = \frac{2}{3}\log\frac{E}{E_0}$$

$$M = \frac{2}{3}\log\frac{1.27 \times 10^{15}}{10^{4.4}}$$

$$M = \frac{2}{3}\log(1.27 \times 10^{10.6})$$

$$M \approx \frac{2}{3}(10.704)$$

$$M \approx 7.136$$

Properties of Logarithms

Properties of Logarithms

- If b, M, and N are positive real numbers, where $b \neq 1$ and p and x are real numbers, then the following are true:
  1. $\log_b 1 = 0$
  2. $\log_b b = 1$
  3. $\log_b b^x = x$
  4. $b^{\log_b x} = x \quad x > 0$
  5. Product Rule: Log of a product is the sum of the logs.

  $$\log_b MN = \log_b M + \log_b N$$

  6. Quotient Rule: Log of a quotient is the difference of the logs.

  $$\log_b \left(\frac{M}{N}\right) = \log_b M - \log_b N$$

  7. Power Rule: Log of a number raised to an exponent is the exponent times the log of the number.

  $$\log_b M^p = p \log_b M$$

- **Example 1:** Use the properties of logs to simplify the following expressions.

  a) $\log 1 - \log 1000^x$

  Since $\boldsymbol{\log x}$ has base **10**, we view this expression as $\boldsymbol{\log_{10} 1 - \log_{10} 1000^x}$. Then use properties 1 and 3 to simplify the expression.

  $$\log 1 - \log 1000^x = 0 - \log(10^3)^x = -\log 10^{3x} = -3x$$

  b) $e^{-3\ln 2}$

  Use properties of exponents first, then use property 4 of logarithms.

$$e^{-3\ln 2} = \left(e^{\ln 2}\right)^{-3} = \frac{1}{(e^{\ln 2})^3} = \frac{1}{2^3} = \frac{1}{8}$$

c) $\log_6 \frac{24}{72}$

Use the quotient, product, and power rules first, then simplify using properties 2 and 3.

$$\log_6 \frac{18}{108} = \log_6 18 - \log_6 108 = \log_6 6 + \log_6 3 - \log_6 6^2 - \log_6 3$$
$$= 1 + \log_6 3 - 2\log_6 6 - \log_6 3 = 1 - 2 + \log_6 3 - \log_6 3 = -1$$

- **Example 2:** Write $\frac{1}{4}\ln(x^2+3) - \frac{1}{3}\ln(x^3-5) + \ln(x+2)$ as a single logarithm.

Use the power property on the first and second terms.	$= \ln(x^2+3)^{1/4} - \ln(x^3-5)^{1/3} + \ln(x+2)$
Use the quotient property on the first and second terms.	$= \ln\frac{(x^2+3)^{1/4}}{(x^3-5)^{1/3}} + \ln(x+2)$
Use the product property.	$= \ln\left[\frac{(x+2)(x^2+3)^{1/4}}{(x^3-5)^{1/3}}\right]$

- **Example 3:** Write $\log\left[\frac{x^2+8x-9}{x^2-4x-12}\right]$ as the sum or difference of logarithms.

Factor the numerator and denominator.	$= \log\left[\frac{(x+9)(x-1)}{(x+2)(x-6)}\right]$
Use the quotient property.	$= \log[(x+9)(x-1)] - \log[(x+2)(x-6)]$
Use the product property.	$= \log(x+9) + \log(x-1) - [\log(x+2) + \log(x-6)]$
Eliminate brackets.	$= \log(x+9) + \log(x-1) - \log(x+2) - \log(x-6)$

Change-of-Base Formula

- For any logarithmic bases a and b and any positive number M, the change-of-base formula says that

$$\log_b M = \frac{\log_a M}{\log_a b}$$

- **Example 1:** Use the change-of-base formula to evaluate $\log_5 26$. Round to four decimal places.

Use the change-of-base formula where $a = 10$.	$\log_5 26 = \frac{\log 26}{\log 5}$
Approximate with a calculator	≈ 2.024369199
	≈ 2.0243

- **Example 2:** Use the change-of-base formula to evaluate $\log_{\pi} e$. Round to four decimal places.

  Use the change-of-base formula where $a = e$.
  $$\log_{\pi} e = \frac{\ln e}{\ln \pi} = \frac{1}{\ln \pi}$$

  Approximate with a calculator

$\approx .8735685268$

$\approx .873$