Exponential and Logarithmic Equations

Exponential Equations

- The following are true when $b > 0$ and $b \neq 1$.
  1. **One-to-one properties:**
     a. $\boldsymbol{b^x = b^y}$ when $\boldsymbol{x = y}$
     b. $\boldsymbol{\log_b x = \log_b y}$ when $\boldsymbol{x = y}$
  2. **Inverse Properties**
     a. $\boldsymbol{b^{\log_b x} = x}$ when $\boldsymbol{x > 0}$
     b. $\boldsymbol{\log_b b^x = x}$

- **Example 1:** Use the one-to-one properties to solve the following exponential equations.

a) $4^x = 64$

Rewrite both sides of the equation in terms of the same base.	$4^x = 4^3$
Use the one-to-one property to identify x.	$x = 3$

b) $6^{10-4x} = 36$

Rewrite both sides of the equation in terms of the same base.	$6^{10-4x} = 6^2$
Use the one-to-one property.	$10 - 4x = 2$
Solve for x.	$x = 2$

c) $\left(\frac{1}{5}\right)^{3y} = 125$

Use the negative-integer exponent property. $\left(\text{i.e. } \frac{1}{5} = 5^{-1}\right)$	$5^{-3y} = 125$
Rewrite both sides of the equation in terms of the same base.	$5^{-3y} = 5^3$
Use the one-to-one property.	$-3y = 3$
Solve for y.	$y = -1$

- **Example 2:** Use the inverse properties to solve the following exponential equations. Round to four decimal places.

  a) $10^{3x+5} = 64$

  Take the common logarithm of both sides. $\log 10^{3x+5} = \log 64$

  Use the inverse property. $3x + 5 = \log 64$

  Solve for x. $x = \dfrac{\log 64 - 5}{3} \approx -1.0646$

  b) $6^{5x-3} = 17$

  Take the logarithm with base 6 of both sides. $\log_6 6^{5x-3} = \log_6 17$

  Use the inverse property. $5x - 3 = \log_6 17$

  Solve for x. $x = \dfrac{\log_6 17 + 3}{5}$

  Use the change-of-basis formula, $\log_6 17 = \frac{\ln 17}{\ln 6}$. $x = \dfrac{\frac{\ln 17}{\ln 6} + 3}{5}$

  Use a calculator to approximate x to four decimal places. $x \approx .9162$

  c) $e^{2x} - 6e^x + 8 = 0$

  Let $u = e^x$. (Note: $u^2 = e^x \cdot e^x = e^{2x}$.) $u^2 - 6u + 8 = 0$

  Factor the trinomial. $(u - 4)(u - 2) = 0$

  Solve for u. $u = 4$ or $u = 2$

  Substitute $u = e^x$. $e^x = 4$ or $e^x = 2$

  Take the natural logarithm of both sides. $\ln e^x = \ln 4$ or $\ln e^x = \ln 2$

  Use the inverse property. $x = \ln 4$ or $x = \ln 2$

  Approximate the right sides. $x \approx 1.3863$ or $x \approx .6931$

Logarithmic Equations

- **Examples:** Use the one-to-one properties to solve the following logarithmic equations. Round to four decimal places.

a) $\log_3(4x - 10) = \log_3(x) + \log_3(x - 3)$

Apply the product property on the right side. $\log_3(4x - 10) = \log_3[x(x - 3)]$

Use the one-to-one property. $4x - 10 = x(x - 3)$

Distribute and simplify. $x^2 - 7x + 10 = 0$

Factor. $(x - 5)(x - 2) = 0$

Solve for x. $x = 5$ or $x = 2$

Eliminate $x = -1$ because $\log_3(-1)$ is undefined.

$$x = 2 : \log_3(2) = \log_3(2) + \overbrace{\log_3(-1)}^{undefined}$$

$$x = 5 : \log_3(10) = \log_3(5) + \log_3(2)$$

$$= \log_3[5(2)] = \log_3(10) \checkmark$$

b) $\log_2(8x) - \log_2(x - 6) = 5$

Use the quotient property on the left side. $\log_2\left(\frac{8x}{x-6}\right) = 5$

Write the equation in exponential form: $\mathbf{\log_b x = y \Rightarrow x = b^y}$. $\frac{8x}{x-6} = 2^5$

Multiply the equation by: $x - 6$. $8x = 2^5(x - 6)$

Simplify the right side. $8x = 32x - 192$

Solve for x. $-24x = -192$

$$x = 8$$

Check. $\log_2[8 \cdot 8] - \log_2[8 - 6]$

$$= \log_2[64] - \log_2[2] = 6 - 1 = 5 \checkmark$$

c) $\ln(5 - x^2) = 3$

Write in exponential form. $5 - x^2 = e^3$

Simplify. $x^2 = 5 - e^3$

No real solution since $5 - e^3$ is negative. $x^2 =$ negative real number

Applications

- **Example 1:** If money is invested in a savings account earning 6.8% interest compounded yearly, how many years will pass until the money quadruples? Round to the nearest year.

  Recall the compound interest formula. $A = P\left(1 + \frac{r}{n}\right)^{nt}$

  Substitute $A = 4P$, $r = 0.068$, and $n = 1$. $4P = P\left(1 + \frac{0.068}{1}\right)^{1 \cdot t}$

  Simplify. $4P = P(1.068)^t$

  Divide both sides by P. $4 = (1.068)^t$

  Take the natural logarithm of both sides. $\ln 4 = \ln(1.068^t)$

  Use the power rule. $\ln 4 = t \ln(1.068)$

  Solve for t. $t = \frac{\ln 4}{\ln(1.068)} \approx 21.072229$

  It will take 21 years for the money to quadruple.

- **Example 2:** If someone has a car horn with a sound intensity of 120 dB, how many watts per square meter does the car horn emit?

  Recall the definition of a decibel. $D = 10 \log\left(\frac{I}{I_T}\right)$

  Substitute $D = 120$, $I_T = 1 \times 10^{-12}\ W/m^2$. $120 = 10 \log\left(\frac{I}{1 \times 10^{-12}}\right)$

  Divide both sides by 10. $12 = \log\left(\frac{I}{1 \times 10^{-12}}\right)$

  Write the equation in exponential form. $10^{12} = \frac{I}{1 \times 10^{-12}}$

  Solve for I. $I = \frac{10^{12}}{10^{12}} = 1\ W/m^2$

  The car horn emits 1 watt per square meter.

Exponential and Models

Exponential Growth Models

- An exponential growth model generally has the form:

$$\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{ce^{kx}},\ \boldsymbol{k} > 0,$$

where $\boldsymbol{c}$ is the current size of the function (usually the value of $\boldsymbol{f}(\boldsymbol{x})$ when $\boldsymbol{x} = 0$), $\boldsymbol{k}$ is the growth rate, and $\boldsymbol{x}$ corresponds to an amount of time.
For example, continuously compounded interest is modelled by an exponential growth model ($A = Pe^{rt}$). Additionally, population growth can be modelled using exponential growth.

  - **Example:** If a colony of 300 bacteria is growing exponentially at a rate of 18% per hour, use the formula $N = N_0e^{rt}$ to find how many bacteria should be in the colony in 12 hours.

    Substitute $N_0 = 300$, $r = .18$, and $t = 12$. $\quad N = 300e^{0.18\cdot 12}$

    Solve for N. $\quad N \approx 2601.3413$

    After 12 hours the population of the bacteria colony will be 2601.

Exponential Decay Models

- An exponential decay model generally has the form:

$$\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{ce^{-kx}}, \boldsymbol{k} > 0,$$

where $\boldsymbol{c}$ is the current size of the function (usually the value of $\boldsymbol{f}(\boldsymbol{x})$ when $\boldsymbol{x} = 0$), $\boldsymbol{k}$ is the growth rate, and $\boldsymbol{x}$ corresponds to an amount of time. Notice that an exponential decay model is identical to exponential growth except $\boldsymbol{k}$ is negative for exponential decay.
One of the most common uses for an exponential decay model is to model radioactive decay. The formula for radioactive decay is: $m = m_0e^{-rt}$ where m_0 represents the initial mass at time $t = 0$, r is the decay rate, t is the time in years, and m is the mass at time t. Typically, the decay rate r is expressed in terms of half-life h where half-life is the time it takes for a quantity to decrease by half. The formula for that is:

$$\boldsymbol{r} = \frac{\ln 2}{\boldsymbol{h}}$$

  - **Example:** If there was 450g of promethium $^{145}\mathrm{Pm}$, how much would be left after 63 years if it has a half-life of 17.7 years? Use the equation $m = m_0e^{-rt}$.
    Find r.

$$\boldsymbol{r} = \frac{\ln 2}{\boldsymbol{h}} = \frac{\ln 2}{\boldsymbol{17.7}} \approx \boldsymbol{.03916}$$

Now, use the formula.

$$m = m_0 e^{-rt}$$
$$m = 450e^{-(.03916)(63)}$$
$$m \approx 38.17\text{mg}$$